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TECHNICAL NOTE

Waverider-wavestaff comparison

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Abstract - The time-history of the sea surface elevation at a fixed horizontal position differs from the time history of the elevation of a specific surface water particle. This leads to a difference in spectral density function obtained from a wavestaff and a free floating buoy. This discrepancy is, in first approximation, calculated and the results are compared with the observed discrepancies between Waverider and wavestaff measurements in the WADIC project.

Introduction

In the paper by Alleder *et al.* (1989), a comparison is given between wave measurement systems.
In this article it is mentioned that at high sea states it is noticed that "above 0.3 Hz the Waverider tends to underestimate the spectral energy".
The results of the measurements were compared with fixed sensors.
It is well known that fixed wave sensors introduce phase modulation which can lead to a substantial "overestimating" of wave energy in the high frequency range at high sea states. This is due to the fact that these non-linear effects create energy at other frequencies than those under consideration.
This can be visualised as follows.
A wave with an amplitude of 5 m and a period of 10 seconds has an orbital speed equal to the phase speed of a 0.5 Hz wave.
So, when both are present, a fixed observer looking at the 0.5 Hz wave "sees" this wave as a 1 Hz wave when maximum horizontal orbital speed is with wave direction and as a 0 Hz wave when maximum orbital speed is against waves. Thus the wave energy belonging to the 0.5 Hz wave is spread out between 0 and 1 Hz. An observer moving with the water particle does not suffer from this spreading effect and measures the spectral density function of a water particle.
The fixed observer measures the spectral density function of the surface at a fixed place. The question “what is best” is irrelevant but one should not compare systems which measure different phenomena.
What can be said in favour of moving observers (buoys) is that one can calculate from the data, when wave directions are also known, what spectral density function will be measured by a fixed observer which is not possible in the opposite direction.
A buoy which is not able to follow completely the horizontal wave movements due to mooring stiffness or protrusions comparable with penetration depth will behave more or less like a fixed observer.

An estimate of the effects of narrow band phase modulation in a wavestaff measuring system

Consider a surface water particle with a rest position at $x = x_p$ (horizontal position). Assume a fixed observer at $x = x_w$, as in Fig. 1. Let x_{tp} be the total horizontal deflection at time t and y_{tp} the total vertical deflection of this water particle. Then the surface water particle observed at time t at x_w has its rest position x_p at

$$x_p = x_w - x_{tp} \qquad (1)$$

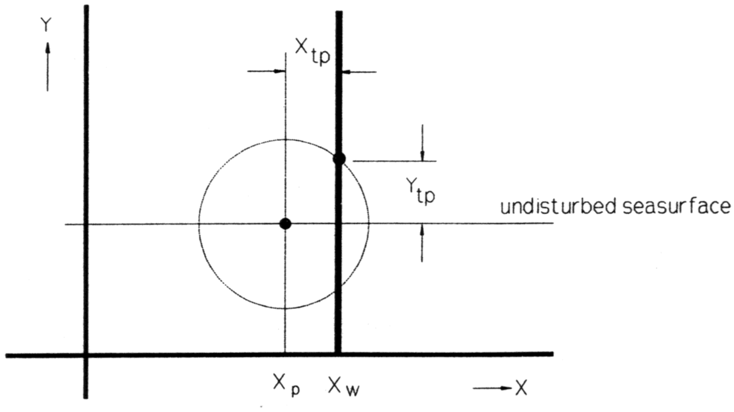

FIG. 1

Assume that the total motion of one specific water particle can be represented by the sum of a great number of sinusoidal motions. That is,

$$x_{tp} = \sum_{n=1}^{\infty} A_n \sin\left(\omega_n t + \phi_n + \frac{2\pi\, x_p}{\lambda_n}\right) \tag{2}$$

and

$$y_{tp} = \sum_{n=1}^{\infty} A_n \cos\left(\omega_n t + \phi_n + \frac{2\pi\, x_p}{\lambda_n}\right) \tag{3}$$

in which ω_n, φ_n and λ_n are independent on the rest position x_p of the water particle under consideration and also independent on time. ω_n is the angular frequency, φ_n is phase and λ_n is the wave length. A_n is the amplitude.

Suppose x_w = zero and $2\pi\, x_{tp} / \lambda_n << 1$. The latter condition is not necessarily fulfilled for high wave frequencies and high sea state because x_{tp} is the total displacement in x direction so that $2\pi\, x_{tp} / \lambda$ can be much greater than the wave slope.

Then y_{tp} as seen by a fixed observer at $x = x_w = 0$ is in first approximation given by:

$$y_{tp} = \sum_{n=1}^{\infty} A_n \cos\left(\omega_n t + \phi_n - \frac{2\pi\, x_{tp}}{\lambda_n}\right) \tag{4}$$

$$\approx \sum_{n=1}^{\infty} A_n \cos\left(\omega_n t + \phi_n\right) + \left[\sum_{n=1}^{\infty} \frac{2\pi\, A_n}{\lambda_n} \sin\left(\omega_n t + \phi_n\right)\right] x_{tp} \tag{5}$$

The term between square brackets is the momentary wave slope which is much smaller than unity so that, when a first-order approximation is required, for x_{tp} the zero-order approximation can be substituted:

$$x_{tp} = \sum_{m=1}^{\infty} A_m \sin\left(\omega_m t + \phi_m\right). \tag{6}$$

Further, for deep water, the substitution

$$\frac{2\pi\, A_n}{\lambda_n} = \frac{\omega_n^2 A_n}{g} \tag{7}$$

can be made, where $\omega_n^2 A_n$ = acceleration = a_n. This results in:

$$y_{tp} = \sum_{n=1}^{\infty} A_n \cos\left(\omega_n t + \phi_n\right) + \left(\sum_{n=1}^{\infty} \frac{\omega_n^2 A_n}{g} \sin\left(\omega_n t + \phi_n\right) \right) \left(\sum_{m=1}^{\infty} A_m \sin\left(\omega_m t + \phi_m\right) \right) \tag{8}$$

The first term is what is measured by a system travelling with a water particle; the second term is the extra due to the fact that the measuring system is at a fixed place. We calculate the extra spectral density following the usual method of Rice found in the collection edited by Wax (1954). Let $\rho_a(f_n)$ be the spectral density for acceleration at frequency f_n and $\rho_A(f_n)$ the spectral density for displacement at frequency f_n. Also,

$$a = \omega^2 A \qquad \omega_n^2 A_n = \sqrt{\left(2\, \rho_a(f_n) df\right)} \qquad A_n = \sqrt{\left(2\, \rho_A(f_n) df\right)}.$$

Consider from both series, the *"m"* and *"n"* series, the *k*th and *l*th term. The *"k,l"* modulation product is

$$\begin{aligned} &\tfrac{1}{2g}\left(\omega_k^2 A_k A_l + \omega_l^2 A_l A_k\right) \; \left[\cos\left((\omega_k - \omega_l)t + \phi_k - \phi_l\right) - \cos\left((\omega_k + \omega_l)t + \phi_k + \phi_l\right)\right] \\ &= \tfrac{1}{g}\left(\sqrt{\left(\rho_a(f_k)\, \rho_A(f_l)\right)} + \sqrt{\left(\rho_a(f_l)\, \rho_A(f_k)\right)}\right)\left[\cos\left((\omega_k - \omega_l)t + \phi_k - \phi_l\right)\right. \\ &\left. - \cos\left((\omega_k + \omega_l)t + \phi_k + \phi_l\right)\right] df \end{aligned} \tag{9}$$

The frequency bands which create modulation products of the sum type in a frequency band 2 *df* near $f_{(+)}$ are lying symmetrically around 0.5 $f_{(+)}$ in the frequency range f = 0 *to* $f_{(+)}$.
The frequency bands creating modulation products of the difference type in a frequency band 2 *df* near $f_{(-)}$ are lying at a frequency distance $f_{(-)}$ of each other.

A pair of frequency bands of width *df* creates energy in a band width of 2 *df* but because the latter band also receives energy from the two adjacent creating bands it is justified, if spectral functions are not too steep, to consider the whole energy created by one couple to be received by one band of total width *df*. We denote the extra energy in band *df* at f_1 due to the difference type as $E_{(-1)}$ and due to the sum type as $E_{(+1)}$ and introduce further an analog notation for the extra spectral density ρ respectively $\rho_{(-1)}$ and $\rho_{(+1)}$. *d*( ) refers to the increment due to a couple of bands with width *df*.

$$d\,\rho_{(+1)} = \frac{dE_{(+1)}}{df} = \frac{1}{2}\left[\frac{df}{g}\left(\sqrt{\left(\rho_a(f)\rho_A(f_1 - f)\right)} + \sqrt{\left(\rho_a(f_1 - f)\rho_A(f)\right)}\right)\right]^2 / df \tag{10}$$

$$d\,\rho_{(-1)} = \frac{dE_{(-1)}}{df} = \frac{1}{2}\left[\frac{df}{g}\left(\sqrt{\left(\rho_a(f)\rho_A(f + f_1)\right)} + \sqrt{\left(\rho_a(f + f_1)\rho_A(f)\right)}\right)\right]^2 / df \tag{11}$$

For the total extra $\rho_{(+1)}$, integration must be performed from f = 0 to $f = 0.5 f_1$.
For the total extra $\rho_{(-1)}$ the integration range is from f = 0 to $f = \infty$.

Toba (1972, 1973) gives for the high frequency part of the spectrum:
The amplitude squared/Hz is $\rho_A(f)$ is $2\ 10^{-5}\ U_{10} g/f^4$ and hence the acceleration amplitude squared/Hz is $\rho_a(f)$ is 0.031 $U_{10}g$ where g is 9.81 m/sec^2 and U_{10} is the wind speed at 10 m height.
In the Toba formulation U^x is replaced by 0.039 U_{10} (Battjes *et al.* 1987) which is valid for U_{10} around 13 m/sec.

To get a rough but easy analytic estimate, we assume a spectral density function which is zero below some lowest frequency, f_L and behaves as described above for f greater than f_L. To obtain with this caricature a realistic relation between windspeed and significant wave height we could choose for $f_L : f_L = 14g/U_{10}$. For f_l smaller than $2 f_L$, $\rho_{(+1)}$ is zero because one of the generating frequency bands is always zero. If f_1 is greater than $2 f_L$, the integrating range for $\rho_{(+1)}$ is from $f = f_L$ to $f = 0.5f_1$. For $\rho_{(-1)}$ the integrating range is from $f = f_L$ to $f = \infty$. Substituting in (10) and (11) the assumed spectral density function and integrating gives:

$$\left.\begin{matrix}\rho_{(+1)}\\ \rho_{(-1)}\end{matrix}\right\} = 10^{-7} U_{10}^2 \left| \frac{\pm 1}{(f_1 \pm f)^3} - \frac{1}{f^3} - \frac{6(1/f \pm 2/f_1)}{f_1(f_1 \pm f)} \pm \frac{12}{f_1^3} \ln\left(\frac{f_1 \pm f}{f}\right) \right|_{f = f_L}^{f = 0.5 f_1 ;\; f = \infty} \qquad (12)$$

The upper sign refers to $\rho_{(+1)}$ while the lower to $\rho_{(-1)}$. The upper limit for $\rho_{(+1)} = 0.5f_1$ and for $\rho_{(-1)} = f_m$ is ∞. As the expression is zero for both upper limits, the values of $\rho_{(+1)}$ and $\rho_{(-1)}$ can be obtained by changing sign and substituting $f = f_L$ in expression (12).
Further $\rho_{(+1)}$ and $\rho_{(-1)}$ are divided by the original spectral density at $f = f_l$ is $\rho_A(f_1) = 2\ 10^{-5} U_{10} g/f_1^4$ in which expression U_{10} is substituted by $0.14g/f_L$. $\rho_A(f_1)$ is what is measured by a moving observer (buoy) and therefore further on denoted by $\rho_B(f_1)$.

$$\left.\begin{matrix}\rho_{(+1)}\\ \rho_{(-1)}\end{matrix}\right\} / \rho_B(f_1) = 7.3\ 10^{-4}\, x^4 \left[1 \pm \left(\frac{1}{x \pm 1}\right)^3 + \frac{6(1 \pm 2/x)}{x(x \pm 1)} \pm \frac{12}{x^3} \ln(x \pm 1) \right] \qquad (13)$$

$$\rho_{(+1)} = 0 \quad for\ x < 2 \qquad (14)$$

$$x = f_1 / f_L \qquad (15)$$

The ratio R_{BS} of the spectral density measured by a moving observer (buoy) ρ_B and the density measured by a fixed observer (staff) ρ_S is:

$$R_{BS} = \rho_B / \rho_s = \rho_B / \left(\rho_B + \rho_{(+1)} + \rho_{(-1)}\right) = 1/\left(1 + \left(\rho_{(+1)} + \rho_{(-1)}\right) / \rho_B(f_1)\right) \qquad (16)$$

In the derivation it is assumed that wave direction for low and high frequencies is the same and that directional spreading is small. If this is not the case the R_{BS} will be closer to unity than calculated. At high sea states there is a good chance that one wave field dominates and the derivation is valid.

As already mentioned the total horizontal water particle displacement must be small compared with the smallest wave length λ_m under consideration to ensure a narrow band phase modulation for which the derivation is valid ($2\pi\, x_{tp} / \lambda_m << 1$). So f_m must be chosen so low that let us say the significant (horizontal) wave amplitude A_s is small compared with $\lambda_m = g/(2\pi f^2_m)$. On the other hand f_m must be chosen so high that the contribution to $\rho_{(-1)}$ in the frequency range between $f = f_m$ and $f = \infty$ can be ignored.
The total power m_o is:

$$m_o = \int_{f_L}^{\infty} \rho_A(f)\, df = 0.66\ 10^{-5} U_{10}\, g / f_L^3 = 0.93\ 10^{-6} g^2 / f_L^4 \qquad (17)$$

The significant wave amplitude A_s is:

$$A_s = 2\sqrt{m_0} = 1.9\ 10^{-3}\ g/\ f_L^2 \tag{18}$$

$$R = 2\pi\ A_s\ /\ \lambda_m = \frac{(2\pi)^2\ 1.9\ 10^{-3}\ gf_m^2}{g\ f_L^2} = 0.076\ f_m^2\ /\ f_L^2 \tag{19}$$

If it is required that $R < 0.5$ then:

$$\frac{f_m}{f_L} < 2.6 \tag{20}$$

The second condition is that the contribution to $\rho_{(-1)}$ in the frequency range from $f = f_m$ to $f = \infty$ is small compared with the contribution of the frequency range from $f = f_L$ to $f = \infty$. Or, the ratio r between expression (12) with $f = f_m$ and expression (12) with $f = f_L$ must be small. If $f_m/f_L = 2.6$, r is maximum for $f_1 = 0.9\sqrt{f_m / f_L}$ and amounts to:

$$r = 1.46\left(f_L / f_m\right)^3 = 0.08 \tag{21}$$

which is, taking into account other approximations, sufficiently small. As both conditions depend on the ratio f_L/f_m the absolute values of f_L and f_m are not important provided one is not interested in $\rho_{(+1)}$ and $\rho_{(-1)}$ for frequency above 2.6 f_L.

In Fig. 2 the calculated ratio R_{BS} is shown as a function of f_l/f_L. Above $f_l/f_L = 2.6$ the line is dotted to indicate that validity is questionable.

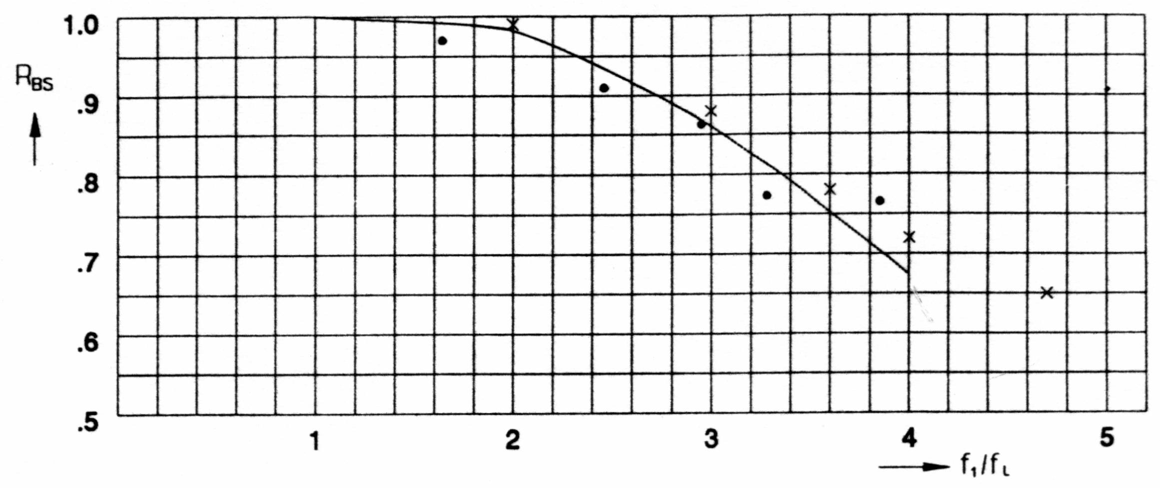

FIG. 2

TABLE 1.

	$2\ m < H_{m0} < 4$ m			$H_{m0} > 4\ m$		
f_1 (Hz)	*WR* Beds	*Wstaff* Beds	*WR* Wstaff	*WR* Beds	*Wstaff* Beds	*WR* Wstaff
0.2	0.97	1	0.97	0.9	0.91	0.99
0.3	1.05	1.15	0.91	0.85	0.97	0.88
0.36	0.95	1.10	0.86	0.78	1.00	0.78
0.4	0.85	1.10	0.77	0.70	0.97	0.72
0.47	0.82	1.07	0.77	0.60	0.93	0.65

R_{BS} is the ratio of the spectral density water particles/spectral density water surface, which is equal to the ratio buoy measurement/wavestaff measurement.
X is the observed ratio Waverider/wavestaff at $H_s > 4\ m$ and $T_L = 10$ sec, and • is the observed ratio Waverider/wavestaff at $2\ m < H_s < 4\ m$ and $T_L = 8.2$ sec. Table 1 is obtained from Allender *et al.* (1989, fig. 10).
WR/BEDS is the ratio between the spectral density obtained from the Waverider and a measuring system BEDS. WSTAFF/BEDS is the same ratio for a WAVESTAFF and BEDS. f_l is the frequency under consideration for which the spectral ratio is given. WR/WSTAFF is the spectral ratio between Waverider and Wavestaff independent on BEDS.

For $H_{m0} > 4\ m$, $T_L = 1/f_L$ is assumed to be 10 sec and for $2\ m < H_{m0} < 4\ m$, T_L is assumed to be 8.2 sec.

Both values are induced by the T_p distribution given in fig. 7 of Allender *et al.* (1989).

In Fig. 2, X is the ratio WR/Wstaff for $H_{m0} > 4\ m$, and • is the ratio WR/Wstaff for $2\ m < H_{m0} < 4\ m$.

The remarkably good agreement with calculated values must be fortuitous because the reading of fig. 10 in Allender *et al.* (1989) was not as accurate as pretended by the notation of Table 1.

For $1\ m < H_{m0} < 2\ m$ the spreading around the calculated values is greater but this was expected. However at this moderate sea state the ratio in the whole frequency range was 10% too low, which is strange given the good agreement at high sea states.

Conclusions

1. The so-called underestimation of wave energy by the Waverider is induced by the fact that a wavestaff distorts the time history of the vertical motion of a water particle
2. It seems that expression (16) gives a reasonable estimate of the ratio buoy/wavestaff measurement up to $f_I/f_L = 4$.
3. Buoys with poor water particle following capabilities in horizontal directions, for instance due to mooring configuration or protrusions not small compared with wave penetration depth will suffer more or less from the same tendency to distort the time history of water particle motion.

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